A triple product identity

Jasper Ty

This document is a write-up of my solution to a homework problem in the class notes ([24ac]) for Math 531, Winter 2024 at Drexel. The problem is

Exercise A.3.3.3 (a) $\boxed{6}$ Prove that

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2} \qquad \text{in } K[[x]]$$

This identity is used in a proof of the Lagrange-Jacobi four-square theorem. [Sambal22]

1 Preliminaries

We adopt all relevant notation from [24ac]. K will denote a general (commutative) ring, $K[z^{\pm}]$ will denote the ring of Laurent polynomials in the indeterminate z over K, and we use $K[[x]]$ to denote the ring of formal power series in the indeterminate x over K.

We use the following important theorem

Theorem: Jacobi Triple Product

In the FPS ring $\mathbb{Z}[z^{\pm}][[q]]$, we have

$$\prod_{n=1}^{\infty}\left(1+q^{2n-1}z\right)\left(1+q^{2n-1}z^{-1}\right)\left(1-q^{2n}\right) = \sum_{\ell\in\mathbb{Z}} q^{\ell^2}z^{\ell}$$

Proof. See section 4.3.3 in [24ac] □

We won't explicitly mention this, but the following fact will lurk in the background

Lemma

Let u, v be units in K, and let a, b be integers such that $a > 0$, $a \geq |b|$. In a purely formal way, define $q^s z^t := (u^s v^t)x^{sa+tb}$. Then the Jacobi triple product holds in $K[[x]]$ in the same form as above.

This says we can "let" $q = ux^a$ and $z = vx^b$ safely.

2 Proof of the identity

An initial attempt can be made by substituting into the Jacobi Triple Product the values $q = x$ and $z = -x$, and we obtain

$$\prod_{n=1}^{\infty}\left(1+x^{2n-1}(-x)\right)\left(1+x^{2n-1}(-x)^{-1}\right)\left(1-x^{2n}\right) = \sum_{\ell\in\mathbb{Z}} x^{\ell^2}(-x)^{\ell}$$

This simplifies to

$$\prod_{n=1}^{\infty}\left(1-x^{2n}\right)^2\left(1-x^{2n-2}\right) = \sum_{\ell\in\mathbb{Z}}(-1)^{\ell}x^{\ell^2+\ell}$$

Which seems ripe for a re-indexing into the form we want. However, on the left hand side, we have a 0 factor, as $1 - x^{2n-2} = 0$ when $n = 1$, and on the right hand side, we have cancellation, as, for all integers k we have $k^2 + k = (-k-1)^2 + (-k-1)$, hence the terms corresponding to $\ell = k$ and $\ell = -k-1$ cancel out. We are left with the identity

$$0 = 0$$

which tells us nothing. (Anything implies a true statement!)

So we try again, but this time we hold off on making our substitutions and deal we with the problematic $(1 + qz^{-1})$ term in the Jacobi Triple Product in $\mathbb{Z}[z^{\pm}][[q]]$ first.

Theorem: Diminished Jacobi Triple Product

In the FPS ring $\mathbb{Z}[z^{\pm}][[q]]$, we have

$$\left(1+qz\right)\left(1-q^2\right)\prod_{n=2}^{\infty}\left(1+q^{2n-1}z\right)\left(1+q^{2n-1}z^{-1}\right)\left(1-q^{2n}\right) = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{2k}(-1)^j q^{k^2+j}z^{k-j}\right]$$

Note that the factor $1 + qz^{-1}$ does not appear on either side.

Proof. This is obtained by dividing both sides of the Jacobi Triple Product by $1 + qz^{-1}$. The left hand side becomes

$$\frac{1}{1+qz^{-1}}\prod_{n=1}^{\infty}\left(1+q^{2n-1}z\right)\left(1+q^{2n-1}z^{-1}\right)\left(1-q^{2n}\right) =$$
$$\left(1+qz\right)\left(1-q^2\right)\prod_{n=2}^{\infty}\left(1+q^{2n-1}z\right)\left(1+q^{2n-1}z^{-1}\right)\left(1-q^{2n}\right) \quad (1)$$

straightforwardly. The right hand side needs more involved computation. First, we note that $1+qz^{-1}$ has a bona-fide inverse in $\mathbb{Z}[z^{\pm}][[q]]$

$$\begin{aligned}(1+qz^{-1})^{-1} &= 1 - q^1z^{-1} + q^2z^{-2} \pm \cdots \\ &= \sum_{i=0}^{\infty}(-1)^i q^i z^{-i}\end{aligned}$$

This tells us that, for all $a \in \mathbb{N}$ and $b \in \mathbb{Z}$

$$\frac{q^a z^b}{1 + q^{2n-1} z^{-1}} = \sum_{i=0}^{\infty} (-1)^i q^{a+i} z^{b-i}$$

Then the right hand side of the Jacobi Triple product divided by $1 + qz^{-1}$ can be written

$$\frac{1}{1 + qz^{-1}} \sum_{\ell \in \mathbb{Z}} q^{\ell^2} z^\ell = \sum_{\ell \in \mathbb{Z}} \sum_{i=0}^{\infty} (-1)^i q^{\ell^2+i} z^{\ell-i}$$

We define, to make calculations nicer, the following FPS $f_{a,b}$

$$f_{a,b} := \frac{q^a z^b}{1 + q^{2n-1} z^{-1}} = \sum_{i=0}^{\infty} (-1)^i q^{a+i} z^{b-i}$$

Then

$$\frac{1}{1 + qz^{-1}} \sum_{\ell \in \mathbb{Z}} q^{\ell^2} z^\ell = \sum_{\ell \in \mathbb{Z}} f_{\ell^2, \ell}$$

And, surprisingly, there is a nice pattern of cancellation that occurs, which has to do with the fact that consecutive perfect squares differ by odd numbers. First we break the outer sum into a nonnegative and negative piece, and re-index both so their summation index runs from 0 to ∞, then recombine them

$$\begin{aligned}
\sum_{\ell \in \mathbb{Z}} f_{\ell^2, \ell} &= \left[\sum_{\ell \geq 0} f_{\ell^2, \ell} \right] + \left[\sum_{\ell < 0} f_{\ell^2, \ell} \right] \\
&= \left[\sum_{k=0}^{\infty} f_{k^2, k} \right] + \left[\sum_{k=0}^{\infty} f_{(-k-1)^2, (-k-1)} \right] \\
&= \left[\sum_{k=0}^{\infty} f_{k^2, k} + f_{(-k-1)^2, (-k-1)} \right]
\end{aligned}$$

Then, we note that we may break up $f_{k^2,k}$ as follows

$$\begin{aligned}
f_{k^2,k} &= \sum_{i=0}^{\infty}(-1)^i q^{k^2+i} z^{k-i} \\
&= \left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right] + \underbrace{\left[\sum_{i=2k+1}^{\infty}(-1)^i q^{k^2+i} z^{k-i}\right]}_{\text{re-index}} \\
&= \left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right] + \left[\sum_{i=0}^{\infty}\underbrace{(-1)^{2k+1+i}}_{=(-1)^{i+1}}\underbrace{q^{k^2+(2k+1+i)}}_{=q^{(-k-1)^2+i}}\underbrace{z^{k-(2k+1+i)}}_{=z^{(-k-1)-i}}\right] \\
&= \left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right] + \left[\sum_{i=0}^{\infty}(-1)^{1+i} q^{(-k-1)^2+i} z^{(-k-1)-i)}\right] \\
&= \left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right] - \left[\sum_{i=0}^{\infty}(-1)^{i} q^{(-k-1)^2+i} z^{(-k-1)-i)}\right] \\
&= \left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right] - f_{(-k-1)^2,(-k-1)}
\end{aligned}$$

which we insert back in

$$\begin{aligned}
&\sum_{k=0}^{\infty} f_{k^2,k} + f_{(-k-1)^2,(-k-1)} \\
&\quad= \sum_{k=0}^{\infty}\left[\left(\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right) - f_{(-k-1)^2,(-k-1)} + f_{(-k-1)^2,(-k-1)}\right] \\
&\quad= \sum_{k=0}^{\infty}\left[\sum_{i=0}^{2k}(-1)^i q^{k^2+i} z^{k-i}\right]
\end{aligned}$$

which was the desired formula for the right hand side □

Then, we attempt the $q = x$, $z = -x$ substitution again, obtaining

Theorem

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2} \qquad \text{in } K[[x]]$$

Proof. By the previous theorem, if we let $q = x$ and $z = -x$, we get for the right hand side

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^j q^{k^2+i} z^{k-j} &= \sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^j x^{k^2+j}(-x)^{k-j} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k x^{k^2+k} \\
&= \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k^2+k} \\
&= \sum_{k=0}^{\infty}(-1)^k(2k+1)(x^2)^{k^2+k/2}
\end{aligned}$$

For the left hand side,

$$\begin{aligned}
&\Big(1+qz\Big)\Big(1-q^2\Big)\prod_{n=2}^{\infty}\Big(1+q^{2n-1}z\Big)\Big(1+q^{2n-1}z^{-1}\Big)(1-q^{2n}) \\
&\qquad= (1-x^2)(1-x^2)\prod_{n=2}^{\infty}(1-x^{2n-2})(1-x^{2n})(1-x^{2n}) \\
&\qquad= (1-x^2)^2\prod_{n=2}^{\infty}(1-x^{2n})^2(1-x^{2n-2}) \\
&\qquad= (1-x^2)^2\prod_{n=2}^{\infty}(1-x^{2n})\prod_{n=2}^{\infty}(1-x^{2n-2})^2 \\
&\qquad= \prod_{n=1}^{\infty}(1-x^{2n})^2\prod_{n=2}^{\infty}(1-x^{2n-2}) \\
&\qquad= \prod_{n=1}^{\infty}(1-x^{2n})\prod_{n=1}^{\infty}(1-x^{2n})^2 \\
&\qquad= \prod_{n=1}^{\infty}(1-x^{2n})^3 \\
&\qquad= \prod_{n=1}^{\infty}(1-(x^2)^n)^3
\end{aligned}$$

So in all we have shown

$$\prod_{n=1}^{\infty}(1-(x^2)^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)(x^2)^{k^2+k/2}$$

and we may "unsubstitute" (Lemma 4.3.3 in [24ac]) x^2 to obtain

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k^2+k/2}$$

□

References

[24ac] Darij Grinberg, *An Introduction to Algebraic Combinatorics*

[Sambal22] Benjamin Sambale, *An invitation to formal power series*, arXiv:2205.00879v4.